

Monogenic functions in the biharmonic boundary value problem

S. V. Gryshchuk¹ & S. A. Plaksa²

¹, ² Institute of Mathematics,

National Academy of Sciences of Ukraine,

Tereshchenkivska Str. 3, 01601, Kiev, Ukraine,

¹ serhii.gryshchuk@gmail.com, gryshchuk@imath.kiev.ua

² plaksa@imath.kiev.ua

Abstract: We consider a commutative algebra \mathbb{B} over the field of complex numbers with a basis $\{e_1, e_2\}$ satisfying the conditions $(e_1^2 + e_2^2)^2 = 0$, $e_1^2 + e_2^2 \neq 0$. Let D be a bounded domain in the Cartesian plane xOy and $D_\zeta = \{xe_1 + ye_2 : (x, y) \in D\}$. Components of every monogenic function $\Phi(xe_1 + ye_2) = U_1(x, y)e_1 + U_2(x, y)ie_1 + U_3(x, y)e_2 + U_4(x, y)ie_2$ having the classic derivative in D_ζ are biharmonic functions in D , i.e. $\Delta^2 U_j(x, y) = 0$ for $j = 1, 2, 3, 4$. We consider a Schwarz-type boundary value problem for monogenic functions in a simply connected domain D_ζ . This problem is associated with the following biharmonic problem: to find a biharmonic function $V(x, y)$ in the domain D when boundary values of its partial derivatives $\partial V/\partial x$, $\partial V/\partial y$ are given on the boundary ∂D . Using a hypercomplex analog of the Cauchy type integral, we reduce the mentioned Schwarz-type boundary value problem to a system of integral equations on the real axes and establish sufficient conditions under which this system has the Fredholm property.

Keywords: biharmonic equation, biharmonic boundary value problem, biharmonic algebra, biharmonic plane, monogenic function, Schwarz-type boundary value problem, biharmonic Cauchy type integral, Fredholm integral equations.

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1 Introduction

Let D be a bounded domain in the Cartesian plane xOy , and let its boundary ∂D be a closed smooth Jordan curve. Let \mathbb{R} be the set of real numbers.

Consider some boundary value problems for *biharmonic functions* $W: D \rightarrow \mathbb{R}$ which have continuous partial derivatives up to the fourth order inclusively and satisfy the biharmonic equation in the domain D :

$$\Delta^2 W(x, y) \equiv \frac{\partial^4 W(x, y)}{\partial x^4} + 2 \frac{\partial^4 W(x, y)}{\partial x^2 \partial y^2} + \frac{\partial^4 W(x, y)}{\partial y^4} = 0.$$

The *principal biharmonic problem* (cf., e.g., [1, p. 194] and [2, p. 13]) consists of finding a function $W: \overline{D} \rightarrow \mathbb{R}$ which is continuous together with partial derivatives of the first order in the closure \overline{D} of the domain D and is biharmonic in D , when its values and values of its outward normal derivative are given on the boundary ∂D :

$$W(x_0, y_0) = \omega_1(s), \quad \frac{\partial W}{\partial \mathbf{n}}(x_0, y_0) = \omega_2(s) \quad \forall (x_0, y_0) \in \partial D, \quad (1.1)$$

where s is an arc coordinate of the point $(x_0, y_0) \in \partial D$.

In the case where ω_1 is a continuously differentiable function, the principal biharmonic problem is equivalent to the following *biharmonic problem* (cf., e.g., [1, p. 194] and [2, p. 13]) on finding a biharmonic function $V: D \rightarrow \mathbb{R}$ with the following boundary conditions:

$$\begin{aligned} \lim_{(x,y) \rightarrow (x_0,y_0), (x,y) \in D} \frac{\partial V(x, y)}{\partial x} &= \omega_3(s), \\ \lim_{(x,y) \rightarrow (x_0,y_0), (x,y) \in D} \frac{\partial V(x, y)}{\partial y} &= \omega_4(s) \quad \forall (x_0, y_0) \in \partial D, \end{aligned} \quad (1.2)$$

$$\int_{\partial D} \left(\omega_3(s) \cos \angle(\mathbf{s}, x) + \omega_4(s) \cos \angle(\mathbf{s}, y) \right) ds = 0.$$

Here given boundary functions ω_3, ω_4 have relations with given functions ω_1, ω_2 of the problem (1.1), viz.,

$$\begin{aligned} \omega_3(s) &= \omega'_1(s) \cos \angle(\mathbf{s}, x) + \omega_2(s) \cos \angle(\mathbf{n}, x), \\ \omega_4(s) &= \omega'_1(s) \cos \angle(\mathbf{s}, y) + \omega_2(s) \cos \angle(\mathbf{n}, y), \end{aligned}$$

where \mathbf{s} and \mathbf{n} denote unit vectors of the tangent and the outward normal to the boundary ∂D , respectively, and $\angle(\cdot, \cdot)$ denotes an angle between an appropriate vector (\mathbf{s} or \mathbf{n}) and the positive direction of coordinate axis (x or y) indicated in the parenthesis. Furthermore, solutions of the problems (1.1) and (1.2) are related by the equality $V(x, y) = W(x, y) + c$, where $c \in \mathbb{R}$.

A technique of using analytic functions of the complex variable for solving the biharmonic problem is based on an expression of biharmonic functions by the Goursat formula. This expression allows to reduce the biharmonic problem to a certain boundary value problem for a pair of analytic functions. Further, expressing analytic functions via the Cauchy type integrals, one can obtain a system of integro-differential equations in the general case. In the case where the boundary ∂D is a Lyapunov curve, the mentioned system can be reduced to a system of Fredholm equations. Such a scheme is developed (cf., e.g., [3, 4, 5, 6]) for solving the main problems of the plane elasticity theory with using a special biharmonic function which is called the Airy stress function.

Another methods for reducing boundary value problems of the plane elasticity theory to integral equations are developed in [2, 7, 8, 9, 10, 11].

In this paper, for solving the biharmonic problem we develop a method which is based on the relation between biharmonic functions and monogenic functions taking values in a commutative algebra. We use an expression of monogenic function by a hypercomplex analog of the Cauchy type integral. Considering a Schwarz type boundary value problem for monogenic functions that is associated with the biharmonic problem, we develop a scheme of its reduction to a system of Fredholm equations in the case where the boundary of domain belongs to a class being wider than the class of Lyapunov curves.

2 Monogenic functions in a biharmonic algebra associated with the biharmonic equation

V. F. Kovalev and I. P. Mel'nichenko [12] considered an associative commutative two-dimensional algebra \mathbb{B} over the field of complex numbers \mathbb{C} with the following multiplication table for basic elements e_1, e_2 :

$$e_1^2 = e_1, \quad e_2 e_1 = e_2, \quad e_2^2 = e_1 + 2ie_2, \quad (2.1)$$

where i is the imaginary complex unit. Elements e_1, e_2 satisfy the relations

$$(e_1^2 + e_2^2)^2 = 0, \quad e_1^2 + e_2^2 \neq 0, \quad (2.2)$$

thereby, all functions $\Phi(\zeta)$ of the variable $\zeta = x e_1 + y e_2$, which have continuous derivatives up to the fourth order inclusively, satisfy the equalities

$$\Delta^2 \Phi(\zeta) = \Phi^{(4)}(\zeta) (e_1^2 + e_2^2)^2 = 0. \quad (2.3)$$

Therefore, components $U_j: D \rightarrow \mathbb{R}$, $j = \overline{1, 4}$, of the expression

$$\Phi(\zeta) = U_1(x, y) e_1 + U_2(x, y) i e_1 + U_3(x, y) e_2 + U_4(x, y) i e_2 \quad (2.4)$$

are biharmonic functions.

Similarly to [12], by the *biharmonic plane* we call a linear span $\mu := \{\zeta = x e_1 + y e_2 : x, y \in \mathbb{R}\}$ of the elements e_1, e_2 satisfying (2.1). With a domain D of the Cartesian plane xOy we associate the congruent domain $D_\zeta := \{\zeta = x e_1 + y e_2 : (x, y) \in D\}$ in the biharmonic plane μ .

We say that a function $\Phi: D_\zeta \rightarrow \mathbb{B}$ is *monogenic* in a domain D_ζ if at every point $\zeta \in D_\zeta$ there exists the derivative of the function Φ :

$$\Phi'(\zeta) := \lim_{h \rightarrow 0, h \in \mu} (\Phi(\zeta + h) - \Phi(\zeta)) h^{-1}.$$

It is proved in [12] that a function $\Phi: D_\zeta \rightarrow \mathbb{B}$ is monogenic in a domain D_ζ if and only if components $U_j: D \rightarrow \mathbb{R}$, $j = \overline{1, 4}$, of the expression (2.4) are differentiable in the domain D and the following analog of the Cauchy – Riemann conditions is satisfied:

$$\frac{\partial \Phi(\zeta)}{\partial y} = \frac{\partial \Phi(\zeta)}{\partial x} e_2 \quad \forall \zeta = x e_1 + y e_2 \in D_\zeta. \quad (2.5)$$

It is established in [13] that every monogenic function $\Phi: D_\zeta \rightarrow \mathbb{B}$ has derivatives $\Phi^{(n)}(\zeta)$ of all orders n in the domain D_ζ and, therefore, it satisfies the equalities (2.3). At the same time, every biharmonic in D function $U(x, y)$ is the first component $U_1 \equiv U$ in the expression (2.4) of a certain monogenic function $\Phi: D_\zeta \rightarrow \mathbb{B}$ and, moreover, all such functions Φ are found in [13] in an explicit form.

3 Boundary value problem for monogenic functions that is associated with the biharmonic problem

Let Φ_1 be monogenic in D_ζ function having the sought-for function $V(x, y)$ of the problem (1.2) as the first component:

$$\Phi_1(\zeta) = V(x, y) e_1 + V_2(x, y) i e_1 + V_3(x, y) e_2 + V_4(x, y) i e_2.$$

It follows from the condition (2.5) for $\Phi = \Phi_1$ that $\partial V_3(x, y)/\partial x = \partial V(x, y)/\partial y$. Therefore,

$$\Phi'_1(\zeta) = \frac{\partial V(x, y)}{\partial x} e_1 + \frac{\partial V_2(x, y)}{\partial x} i e_1 + \frac{\partial V(x, y)}{\partial y} e_2 + \frac{\partial V_4(x, y)}{\partial x} i e_2 \quad (3.1)$$

and, as consequence, we conclude that the biharmonic problem with boundary conditions (1.2) is reduced to the boundary value problem on finding a monogenic in D_ζ function $\Phi \equiv \Phi'_1$ when values of two components $U_1 = \partial V(x, y)/\partial x$ and $U_3 = \partial V(x, y)/\partial y$ of the expression (2.4) are given on the boundary ∂D_ζ of the domain D_ζ .

As in [14], by the (1-3)-*problem* we shall call the problem on finding a monogenic function $\Phi: D_\zeta \rightarrow \mathbb{B}$ when values of components U_1, U_3 of the expression (2.4) are given on the boundary ∂D_ζ :

$$U_1(x_0, y_0) = u_1(\zeta_0), \quad U_3(x_0, y_0) = u_3(\zeta_0) \quad \forall \zeta_0 := x_0 e_1 + y_0 e_2 \in \partial D_\zeta,$$

where $u_1(\zeta_0) \equiv \omega_3(s)$ and $u_3(\zeta_0) \equiv \omega_4(s)$.

Problems of such a type on finding a monogenic function with given boundary values of two its components were posed by V. F. Kovalev [15] who called them by *biharmonic Schwarz problems*, because their formulations are analogous in a certain sense to the classic Schwarz problem on finding an analytic function of the complex variable when values of its real part are given on the boundary of domain. V. F. Kovalev [15] stated a sketch of reduction of biharmonic Schwarz problems to integro-differential equations with using conformal mappings and expressions of monogenic functions via analytic functions of the complex variable.

In [14], we investigated the (1-3)-problem for cases where D_ζ is either an upper half-plane or a unit disk in the biharmonic plane. Its solutions were

found in explicit forms with using of some integrals analogous to the classic Schwarz integral.

In [16], a certain scheme was proposed for reducing the (1-3)-problem in a simply connected domain with sufficiently smooth boundary to a suitable boundary value problem in a disk with using power series and conformal mappings in the complex plane.

Hypercomplex methods for investigating the biharmonic equation were developed in the papers [17, 18, 19, 20, 21, 22] also.

4 Biharmonic Cauchy type integral

Consider the biharmonic Cauchy type integral

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_{\partial D_\zeta} \varphi(\tau)(\tau - \zeta)^{-1} d\tau \quad (4.1)$$

with a continuous density $\varphi: \partial D_\zeta \rightarrow \mathbb{B}$. The integral (4.1) is a monogenic function in both domains D_ζ and $\mu \setminus \overline{D_\zeta}$.

We use the euclidian norm $\|a\| := \sqrt{|z_1|^2 + |z_2|^2}$ in the algebra \mathbb{B} , where $a = z_1 e_1 + z_2 e_2$ and $z_1, z_2 \in \mathbb{C}$. We use also the modulus of continuity of a function $\varphi: \partial D_\zeta \rightarrow \mathbb{B}$:

$$\omega(\varphi, \varepsilon) := \sup_{\tau_1, \tau_2 \in \partial D_\zeta, \|\tau_1 - \tau_2\| \leq \varepsilon} \|\varphi(\tau_1) - \varphi(\tau_2)\|.$$

Consider a singular integral which is understood in the sense of its Cauchy principal value:

$$\int_{\partial D_\zeta} \varphi(\tau)(\tau - \zeta_0)^{-1} d\tau := \lim_{\varepsilon \rightarrow 0} \int_{\partial D_\zeta \setminus \partial D_\zeta^\varepsilon(\zeta_0)} \varphi(\tau)(\tau - \zeta_0)^{-1} d\tau,$$

where $\zeta_0 \in \partial D_\zeta$, $\partial D_\zeta^\varepsilon(\zeta_0) := \{\tau \in \partial D_\zeta : \|\tau - \zeta_0\| \leq \varepsilon\}$.

The following theorem can be proved in a similar way as an appropriate theorem in the complex plane (cf., e.g., [23, 24]). It presents sufficient conditions for the existence of limiting values

$$\Phi^+(\zeta_0) := \lim_{\zeta \rightarrow \zeta_0, \zeta \in D_\zeta} \Phi(\zeta), \quad \Phi^-(\zeta_0) := \lim_{\zeta \rightarrow \zeta_0, \zeta \in \mu \setminus \overline{D_\zeta}} \Phi(\zeta)$$

of the biharmonic Cauchy type integral in any point $\zeta_0 \in \partial D_\zeta$.

Theorem 4.2 *Let the modulus of continuity of a function $\varphi: \partial D_\zeta \longrightarrow \mathbb{B}$ satisfy the Dini condition*

$$\int_0^1 \frac{\omega(\varphi, \eta)}{\eta} d\eta < \infty. \quad (4.3)$$

Then the integral (4.1) has limiting values $\Phi^\pm(\zeta_0)$ in any point $\zeta_0 \in \partial D_\zeta$ that are represented by the Sokhotski–Plemelj formulas:

$$\begin{aligned} \Phi^+(\zeta_0) &= \frac{1}{2} \varphi(\zeta_0) + \frac{1}{2\pi i} \int_{\partial D_\zeta} \varphi(\tau)(\tau - \zeta_0)^{-1} d\tau, \\ \Phi^-(\zeta_0) &= -\frac{1}{2} \varphi(\zeta_0) + \frac{1}{2\pi i} \int_{\partial D_\zeta} \varphi(\tau)(\tau - \zeta_0)^{-1} d\tau. \end{aligned} \quad (4.4)$$

5 Scheme for reducing the (1-3)-problem to a system of integral equations

Let the functions $u_1: \partial D_\zeta \longrightarrow \mathbb{R}$, $u_3: \partial D_\zeta \longrightarrow \mathbb{R}$ satisfy conditions of the type (4.3).

We shall find solutions of the (1-3)-problem in the class of functions represented in the form

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_{\partial D_\zeta} \left(\varphi_1(\tau)e_1 + \varphi_3(\tau)e_2 \right) (\tau - \zeta)^{-1} d\tau \quad \forall \zeta \in D_\zeta, \quad (5.1)$$

where the functions $\varphi_1: \partial D_\zeta \rightarrow \mathbb{R}$ and $\varphi_3: \partial D_\zeta \rightarrow \mathbb{R}$ satisfy conditions of the type (4.3).

Then, by Theorem 4.2, the following equality is valid for any $\zeta_0 \in \partial D_\zeta$:

$$\Phi^+(\zeta_0) := \frac{1}{2} \left(\varphi_1(\zeta_0)e_1 + \varphi_3(\zeta_0)e_2 \right) + \frac{1}{2\pi i} \int_{\partial D_\zeta} \left(\varphi_1(\tau)e_1 + \varphi_3(\tau)e_2 \right) (\tau - \zeta_0)^{-1} d\tau. \quad (5.2)$$

By D_z we denote the domain in \mathbb{C} which is congruent to the domain D , i.e. $D_z := \{z = x + iy \in \mathbb{C} : (x, y) \in D\}$. We shall use a conformal mapping

$z = \tau(t)$ of the upper half-plane $\{t \in \mathbb{C} : \text{Im } t > 0\}$ onto the domain D_z . Denote $\tau_1(t) := \text{Re } \tau(t)$, $\tau_2(t) := \text{Im } \tau(t)$.

Inasmuch as the mentioned conformal mapping is continued to a homeomorphism between the closures of corresponding domains, the function

$$\tilde{\tau}(s) := \tau_1(s)e_1 + \tau_2(s)e_2 \quad \forall s \in \overline{\mathbb{R}} \quad (5.3)$$

generates a homeomorphic mapping of the extended real axis $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ onto the curve ∂D_ζ .

Introducing the function

$$g(s) := g_1(s)e_1 + g_3(s)e_2 \quad \forall s \in \overline{\mathbb{R}}, \quad (5.4)$$

where $g_j(s) := \varphi_j(\tilde{\tau}(s))$ for $j \in \{1, 3\}$, we rewrite the equality (5.2) in the form

$$\Phi^+(\zeta_0) = \frac{1}{2}g(t) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(s)(\tilde{\tau}(s) - \tilde{\tau}(t))^{-1} \tilde{\tau}'(s) ds \quad \forall t \in \mathbb{R}, \quad (5.5)$$

where the integral is understood in the sense of its Cauchy principal value (cf., e.g., [23]) and a correspondence between the points $\zeta_0 \in \partial D_\zeta \setminus \{\tilde{\tau}(\infty)\}$ and $t \in \mathbb{R}$ is given by the equality $\zeta_0 = \tilde{\tau}(t)$.

To transform the expression under an integral sign in the equality (5.5) we use the equalities

$$(\tilde{\tau}(s) - \tilde{\tau}(t))^{-1} = \frac{1}{\tau(s) - \tau(t)} + \frac{i(\tau_2(s) - \tau_2(t))}{2(\tau(s) - \tau(t))^2} \rho,$$

$$\tilde{\tau}'(s) = \tau'(s) - \frac{i\tau_2'(s)}{2} \rho,$$

where

$$\rho := 2e_1 + 2ie_2 \quad (5.6)$$

is a nilpotent element of the algebra \mathbb{B} because $\rho^2 = 0$. Thus, we transform the equality (5.5) into the form

$$\Phi^+(\zeta_0) = \frac{1}{2}g(t) + \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(s)k(t, s) ds + \frac{1}{2\pi i} \int_{-\infty}^{\infty} g(s) \frac{1+st}{(s-t)(s^2+1)} ds,$$

where

$$k(t, s) = k_1(t, s)e_1 + i\rho k_2(t, s), \quad (5.7)$$

$$k_1(t, s) := \frac{\tau'(s)}{\tau(s) - \tau(t)} - \frac{1 + st}{(s - t)(s^2 + 1)}, \quad (5.8)$$

$$k_2(t, s) := \frac{\tau'(s)(\tau_2(s) - \tau_2(t))}{2(\tau(s) - \tau(t))^2} - \frac{\tau_2'(s)}{2(\tau(s) - \tau(t))}. \quad (5.9)$$

We use the notations $U_j[a] := a_j$, $j = \overline{1, 4}$, where $a_j \in \mathbb{R}$ is the coefficient in the decomposition of element $a = a_1e_1 + a_2ie_1 + a_3e_2 + a_4ie_2 \in \mathbb{B}$ with respect to the basis $\{e_1, e_2\}$.

In order to single out $U_1[\Phi^+(\zeta_0)]$, $U_3[\Phi^+(\zeta_0)]$ we use the equalities (5.4), (5.6), (5.7) and get the decomposition of the following expression with respect to the basis $\{e_1, e_2\}$:

$$\begin{aligned} g(s)k(t, s) &= (g_1(s)e_1 + g_3(s)e_2)(k_1(t, s)e_1 + i(2e_1 + 2ie_2)k_2(t, s)) = \\ &= (g_1(s)(k_1(t, s) + 2ik_2(t, s)) - 2g_3(s)k_2(t, s))e_1 + \\ &+ (g_3(s)(k_1(t, s) - 2ik_2(t, s)) - 2g_1(s)k_2(t, s))e_2. \end{aligned}$$

Now, we single out $U_1[\Phi^+(\zeta_0)]$, $U_3[\Phi^+(\zeta_0)]$ and obtain the following system of integral equations for finding the functions g_1 and g_3 :

$$\begin{aligned} U_1[\Phi^+(\zeta_0)] &\equiv \frac{1}{2}g_1(t) + \frac{1}{2\pi} \int_{-\infty}^{\infty} g_1(s) \left(\operatorname{Im} k_1(t, s) + 2\operatorname{Re} k_2(t, s) \right) ds - \\ &- \frac{1}{\pi} \int_{-\infty}^{\infty} g_3(s) \operatorname{Im} k_2(t, s) ds = \tilde{u}_1(t), \\ U_3[\Phi^+(\zeta_0)] &\equiv \frac{1}{2}g_3(t) - \frac{1}{\pi} \int_{-\infty}^{\infty} g_1(s) \operatorname{Im} k_2(t, s) ds + \\ &+ \frac{1}{2\pi} \int_{-\infty}^{\infty} g_3(s) \left(\operatorname{Im} k_1(t, s) - 2\operatorname{Re} k_2(t, s) \right) ds = \tilde{u}_3(t) \quad \forall t \in \mathbb{R}, \end{aligned} \quad (5.10)$$

where $\tilde{u}_j(t) := u_j(\tilde{\tau}(t))$, $j \in \{1, 3\}$.

Below, we shall state conditions which are sufficient for compactness of integral operators on the left-hand sides of equations of the system (5.10).

6 Auxiliary Statements

For a function $\varphi : \gamma \rightarrow \mathbb{C}$ which is continuous on the curve $\gamma \subset \mathbb{C}$, a modulus of continuity is defined by the equality

$$\omega_\gamma(\varphi, \varepsilon) := \sup_{t_1, t_2 \in \gamma, |t_1 - t_2| \leq \varepsilon} |\varphi(t_1) - \varphi(t_2)|.$$

Consider the conformal mapping $\sigma(T)$ of the unit disk $\{T \in \mathbb{C} : |T| < 1\}$ onto the domain D_z such that $\tau(t) = \sigma\left(\frac{t-i}{t+i}\right)$ for all $t \in \{t \in \mathbb{C} : \operatorname{Im} t > 0\}$. Denote $\sigma_1(T) := \operatorname{Re} \sigma(T)$, $\sigma_2(T) := \operatorname{Im} \sigma(T)$.

Assume that the conformal mapping $\sigma(T)$ has the continuous contour derivative $\sigma'(T)$ on the unit circle $\Gamma := \{T \in \mathbb{C} : |T| = 1\}$ and $\sigma'(T) \neq 0$ for all $T \in \Gamma$. Then there exist constants c_1 and c_2 such that the following inequalities are valid:

$$0 < c_1 \leq \left| \frac{\sigma(S) - \sigma(T)}{S - T} \right| \leq c_2. \quad (6.1)$$

In this case, for all $S, T_1, T_2 \in \Gamma$ such that $0 < |S - T_1| < |S - T_2|$ the following estimates are also valid:

$$\left| \frac{\sigma_j(S) - \sigma_j(T_1)}{S - T_1} - \frac{\sigma_j(S) - \sigma_j(T_2)}{S - T_2} \right| \leq c \frac{\omega_\Gamma(\sigma', |S - T_2|)}{|S - T_2|} |T_1 - T_2|, \quad j = 1, 2, \quad (6.2)$$

where the constant c does not depend on S, T_1, T_2 . Estimates of the same type for the function $\sigma(T)$ are corollaries of the estimates (6.2). They are adduced in [25].

It follows from (6.2) that the inequalities

$$\left| \sigma'_j(S) - \frac{\sigma_j(S) - \sigma_j(T)}{S - T} \right| \leq c \omega_\Gamma(\sigma', |S - T|), \quad j = 1, 2, \quad (6.3)$$

are fulfilled for all $S, T \in \Gamma$, $S \neq T$, where the constant c does not depend on S and T . Then an inequality of the same type for the function $\sigma(T)$ is certainly fulfilled.

Let $C(\overline{\mathbb{R}})$ denote the Banach space of functions $g : \overline{\mathbb{R}} \rightarrow \mathbb{C}$ that are continuous on the extended real axis $\overline{\mathbb{R}}$ with the norm $\|g\|_{C(\overline{\mathbb{R}})} := \sup_{t \in \overline{\mathbb{R}}} |g(t)|$.

Lemma 6.4 *Let $g \in C(\overline{\mathbb{R}})$ and the conformal mapping $\sigma(T)$ have the non-vanishing continuous contour derivative $\sigma'(T)$ on the circle Γ , and its modulus of continuity satisfy the Dini condition*

$$\int_0^1 \frac{\omega_\Gamma(\sigma', \eta)}{\eta} d\eta < \infty. \quad (6.5)$$

Then for $0 < \varepsilon < 1/4$ and $t \in \mathbb{R}$ the following estimates are true:

$$\left| \int_{-\infty}^{\infty} g(s) k_j(t + \varepsilon, s) ds - \int_{-\infty}^{\infty} g(s) k_j(t, s) ds \right| \leq c \|g\|_{C(\overline{\mathbb{R}})} \varepsilon \int_0^2 \frac{\omega_\Gamma(\sigma', \eta)}{\eta(\eta + \varepsilon)} d\eta, \quad j = 1, 2, \quad (6.6)$$

where $\varepsilon := \varepsilon/(t^2 + 1)$ and the constant c does not depend on t and ε .

Proof. Let us prove the inequality (6.6) for $j = 1$. Denote $S := \frac{s-i}{s+i}$, $T := \frac{t-i}{t+i}$, $d(S, T) := \frac{\sigma(S) - \sigma(T)}{S - T}$. Taking into account the equalities

$$S - T = \frac{2i(s - t)}{(s + i)(t + i)}, \quad (6.7)$$

$$\tau'(s) = \frac{2i}{(s + i)^2} \sigma'(S),$$

we transform the expression

$$\frac{\tau'(s)}{\tau(s) - \tau(t)} = \frac{\frac{2i}{(s+i)^2} \sigma'(S)}{(S - T)d(S, T)} = \left(\frac{1 + st}{(s - t)(s^2 + 1)} + i \frac{1}{s^2 + 1} \right) \frac{\sigma'(S)}{d(S, T)}$$

and represent the function $k_1(t, s)$ in the form

$$k_1(t, s) = m_1(t, s) + i m_2(t, s), \quad (6.8)$$

where

$$m_1(t, s) := \frac{1 + st}{(s - t)(s^2 + 1)} \frac{\sigma'(S) - d(S, T)}{d(S, T)}, \quad m_2(t, s) := \frac{\sigma'(S)}{d(S, T)(s^2 + 1)}.$$

Further, representing the integral

$$I[g, k_1](t) := \int_{-\infty}^{\infty} g(s) k_1(t, s) ds \quad \forall t \in \mathbb{R}$$

by the sum of two integrals

$$I[g, k_1](t) = I[g, m_1](t) + iI[g, m_2](t) \quad \forall t \in \mathbb{R}, \quad (6.9)$$

we shall obtain estimates of the type (6.6) for each of integrals $I[g, m_1](t)$, $I[g, m_2](t)$.

For the integral $I[g, m_1](t)$, we have

$$\begin{aligned} \left| I[g, m_1](t + \varepsilon) - I[g, m_1](t) \right| &\leq \int_{t-2\varepsilon}^{t+2\varepsilon} |g(s)| |m_1(t, s)| ds + \\ &+ \int_{t-2\varepsilon}^{t+2\varepsilon} |g(s)| |m_1(t + \varepsilon, s)| ds + \\ &+ \left(\int_{-\infty}^{t-2\varepsilon} + \int_{t+2\varepsilon}^{\infty} \right) |g(s)| |m_1(t, s) - m_1(t + \varepsilon, s)| ds =: J_1 + J_2 + J_3. \end{aligned}$$

Taking into account the relations (6.1), (6.3), (6.7), we obtain

$$\begin{aligned} J_1 &\leq c \|g\|_{C(\mathbb{R})} \int_{t-2\varepsilon}^{t+2\varepsilon} \frac{\omega_{\Gamma}(\sigma', |S - T|)}{|S - T|} \frac{(1 + |s||t|)}{\sqrt{s^2 + 1}\sqrt{t^2 + 1}} \frac{ds}{s^2 + 1} \leq \\ &\leq c \|g\|_{C(\mathbb{R})} \int_{t-2\varepsilon}^{t+2\varepsilon} \frac{\omega_{\Gamma}(\sigma', |S - T|)}{|S - T|} \frac{ds}{s^2 + 1} \leq \\ &\leq c \|g\|_{C(\mathbb{R})} \int_0^{8\varepsilon} \frac{\omega_{\Gamma}(\sigma', \eta)}{\eta} d\eta \leq c \|g\|_{C(\mathbb{R})} \varepsilon \int_0^2 \frac{\omega_{\Gamma}(\sigma', \eta)}{\eta(\eta + \varepsilon)} d\eta. \end{aligned}$$

Here and below in the proof, by c we denote constants whose values are independent of t and ε , but, generally speaking, may be different even within a single chain of inequalities.

The integral J_2 is similarly estimated.

To estimate the integral J_3 , take into consideration the point $T_1 := \frac{t+\varepsilon-i}{t+\varepsilon+i}$. Using the equality

$$\begin{aligned} m_1(t, s) - m_1(t + \varepsilon, s) &= \frac{-\varepsilon}{(s-t)(s-t-\varepsilon)} \frac{\sigma'(S) - d(S, T)}{d(S, T)} + \\ &+ \frac{1 + s(t + \varepsilon)}{(s-t-\varepsilon)(s^2 + 1)} \frac{(\sigma'(S) - d(S, T))(d(S, T_1) - d(S, T))}{d(S, T)d(S, T_1)} + \\ &+ \frac{1 + s(t + \varepsilon)}{(s-t-\varepsilon)(s^2 + 1)} \frac{d(S, T_1) - d(S, T)}{d(S, T_1)}, \end{aligned}$$

we estimate J_3 by the sum of three integrals:

$$\begin{aligned} J_3 &\leq \varepsilon \left(\int_{-\infty}^{t-2\varepsilon} + \int_{t+2\varepsilon}^{\infty} \right) |g(s)| \frac{|\sigma'(S) - d(S, T)|}{|d(S, T)||s-t||s-t-\varepsilon|} ds + \\ &+ \left(\int_{-\infty}^{t-2\varepsilon} + \int_{t+2\varepsilon}^{\infty} \right) |g(s)| |d(S, T_1) - d(S, T)| \times \\ &\times \frac{|\sigma'(S) - d(S, T)|(1 + |s||t + \varepsilon|)}{|d(S, T)||d(S, T_1)||s-t-\varepsilon|} \frac{ds}{s^2 + 1} + \\ &+ \left(\int_{-\infty}^{t-2\varepsilon} + \int_{t+2\varepsilon}^{\infty} \right) |g(s)| \frac{|d(S, T_1) - d(S, T)|(1 + |s||t + \varepsilon|)}{|d(S, T_1)||s-t-\varepsilon|} \frac{ds}{s^2 + 1} =: \\ &=: \sum_{j=1}^3 J_{3,j}. \end{aligned}$$

Taking into account the inequalities $|s-t| \leq 2|s-t-\varepsilon| \leq 3|s-t|$ for all $s \in (-\infty, t-2\varepsilon] \cup [t+2\varepsilon, +\infty)$ and the relations (6.1), (6.3), (6.7), we obtain

$$\begin{aligned} J_{3,1} &\leq c \|g\|_{C(\overline{\mathbb{R}})} \varepsilon \left(\int_{-\infty}^{t-2\varepsilon} + \int_{t+2\varepsilon}^{\infty} \right) \frac{\omega_{\Gamma}(\sigma', |S-T|)}{|s-t|^2} ds = \\ &= c \|g\|_{C(\overline{\mathbb{R}})} \frac{\varepsilon}{t^2 + 1} \left(\int_{-\infty}^{t-2\varepsilon} + \int_{t+2\varepsilon}^{\infty} \right) \frac{\omega_{\Gamma}(\sigma', |S-T|)}{|S-T|^2} \frac{ds}{s^2 + 1} \leq \end{aligned}$$

$$\leq c \|g\|_{C(\overline{\mathbb{R}})} \epsilon \int_{\epsilon}^2 \frac{\omega_{\Gamma}(\sigma', \eta)}{\eta^2} d\eta \leq c \|g\|_{C(\overline{\mathbb{R}})} \epsilon \int_0^2 \frac{\omega_{\Gamma}(\sigma', \eta)}{\eta(\eta + \epsilon)} d\eta.$$

Using the inequalities (6.2) and properties of a modulus of continuity (cf., e.g., [26, p. 176]), we obtain the following inequalities similarly to the estimation of $J_{3,1}$:

$$\begin{aligned} J_{3,2} &\leq c \|g\|_{C(\overline{\mathbb{R}})} \left(\int_{-\infty}^{t-2\epsilon} + \int_{t+2\epsilon}^{\infty} \right) \frac{|d(S, T_1) - d(S, T)| (1 + |s| |t + \epsilon|)}{|s - t - \epsilon|} \frac{ds}{s^2 + 1} \leq \\ &\leq c \|g\|_{C(\overline{\mathbb{R}})} |T_1 - T| \left(\int_{-\infty}^{t-2\epsilon} + \int_{t+2\epsilon}^{\infty} \right) \frac{\omega(\sigma', |S - T|)}{|S - T|} \frac{(1 + |s| |t + \epsilon|)}{|s - t|} \frac{ds}{s^2 + 1} \leq \\ &\leq c \|g\|_{C(\overline{\mathbb{R}})} |T_1 - T| \left(\int_{-\infty}^{t-2\epsilon} + \int_{t+2\epsilon}^{\infty} \right) \frac{\omega(\sigma', |S - T|)}{|S - T|^2} \frac{(1 + |s| |t + \epsilon|)}{\sqrt{s^2 + 1} \sqrt{t_1^2 + 1}} \frac{ds}{s^2 + 1} \leq \\ &\leq c \|g\|_{C(\overline{\mathbb{R}})} \epsilon \left(\int_{-\infty}^{t-2\epsilon} + \int_{t+2\epsilon}^{\infty} \right) \frac{\omega(\sigma', |S - T|)}{|S - T|^2} \frac{ds}{s^2 + 1} \leq \\ &\leq c \|g\|_{C(\overline{\mathbb{R}})} \epsilon \int_0^2 \frac{\omega_{\Gamma}(\sigma', \eta)}{\eta(\eta + \epsilon)} d\eta. \end{aligned}$$

The integral $J_{3,3}$ is similarly estimated. An estimate of the type (6.6) for $I[g, m_1](t)$ follows from the obtained estimates.

By the scheme used above for estimating the integral $I[g, m_1](t)$, we get the estimate

$$\left| I[g, m_2](t + \epsilon) - I[g, m_2](t) \right| \leq c \|g\|_{C(\overline{\mathbb{R}})} \epsilon,$$

whence an estimate of the type (6.6) follows for $I[g, m_2](t)$. Thus, the inequality (6.6) is proved for $j = 1$.

Let us prove the inequality (6.6) for $j = 2$. Denote $d_j(S, T) := \frac{\sigma_j(S) - \sigma_j(T)}{S - T}$ for $j = 1, 2$.

Similarly to the expression (6.8), we represent the function $k_2(t, s)$ in the form

$$k_2(t, s) = n_1(t, s) + in_2(t, s), \quad (6.10)$$

where

$$\begin{aligned}
n_1(t, s) &:= \frac{1 + st}{(s - t)(s^2 + 1)} \times \\
&\times \frac{\sigma'_1(S) \left(d_2(S, T) - \sigma'_2(S) \right) + \sigma'_2(S) \left(\sigma'_1(S) - d_1(S, T) \right)}{2 (d(S, T))^2}, \\
n_2(t, s) &:= \frac{\sigma'_1(S) \left(d_2(S, T) - \sigma'_2(S) \right) + \sigma'_2(S) \left(\sigma'_1(S) - d_1(S, T) \right)}{2 (d(S, T))^2 (s^2 + 1)}.
\end{aligned}$$

Now, estimates of the type (6.6) for $I[g, n_1](t)$, $I[g, n_2](t)$ are established similarly to analogous estimates for $I[g, m_1](t)$, $I[g, m_2](t)$, respectively. The lemma is proved.

Consider the notations

$$\begin{aligned}
k_1(\infty, s) &:= -\frac{s}{s^2 + 1} \frac{\sigma'(S) - d(S, 1)}{d(S, 1)} + i \frac{\sigma'(S)}{(s^2 + 1) d(S, 1)} =: \\
&=: m_1(\infty, s) + im_2(\infty, s), \\
k_2(\infty, s) &:= -\frac{s}{2(s^2 + 1) (d(S, 1))^2} \times \\
&\times \left(\sigma'(S) (d_2(S, 1) - \sigma'_2(S)) + \sigma'_2(S) (\sigma'(S) - d(S, 1)) \right) + \\
&+ i \frac{1}{2(s^2 + 1) (d(S, 1))^2} \left(\sigma'(S) d_2(S, 1) - \sigma'_2(S) d(S, 1) \right) =: \\
&=: n_1(\infty, s) + in_2(\infty, s).
\end{aligned}$$

Lemma 6.11 *Let $g \in C(\overline{\mathbb{R}})$ and the conformal mapping $\sigma(T)$ have the nonvanishing continuous contour derivative $\sigma'(T)$ on the circle Γ , and its modulus of continuity satisfy the condition (6.5). Then for $0 < \varepsilon < 1/4$ and $t \in \mathbb{R}$ such that $|t| > 1/\varepsilon$ the following estimates are true:*

$$\begin{aligned}
&\left| \int_{-\infty}^{\infty} g(s) k_j(t, s) ds - \int_{-\infty}^{\infty} g(s) k_j(\infty, s) ds \right| \leq \\
&\leq c \|g\|_{C(\overline{\mathbb{R}})} \varepsilon \int_0^2 \frac{\omega_{\Gamma}(\sigma', \eta)}{\eta(\eta + \varepsilon)} d\eta, \quad j = 1, 2, \tag{6.12}
\end{aligned}$$

where the constant c does not depend on t and ε .

Proof. Consider the case $t > 1/\varepsilon$ (the case $t < -1/\varepsilon$ is considered by analogy).

In order to prove the estimate (6.12) for $j = 1$, we shall use the expression (6.9) of the integral $I[g, k_1](t)$ and obtain estimates of the type (6.12) for each of the integrals $I[g, m_1](t)$, $I[g, m_2](t)$.

For $I[g, m_1](t)$ we have

$$\begin{aligned} & \left| \int_{-\infty}^{\infty} g(s) m_1(t, s) ds - \int_{-\infty}^{\infty} g(s) m_1(\infty, s) ds \right| \leq \\ & \leq \left(\int_{-\infty}^{-t/2} + \int_{3t/2}^{\infty} \right) |g(s)| \left(|m_1(t, s)| + |m_1(\infty, s)| \right) ds + \\ & \quad + \int_{t/2}^{3t/2} |g(s)| \left(|m_1(t, s)| + |m_1(\infty, s)| \right) ds + \\ & \quad + \int_{-t/2}^{t/2} |g(s)| |m_1(t, s) - m_1(\infty, s)| ds =: I_1 + I_2 + I_3. \end{aligned}$$

The integrals I_1 and I_2 are estimated with using (6.3):

$$\begin{aligned} I_1 & \leq c \|g\|_{C(\overline{\mathbb{R}})} \left(\left(\int_{-\infty}^{-t/2} + \int_{3t/2}^{\infty} \right) \omega_{\Gamma}(\sigma', |S - T|) \frac{t ds}{s^2 + 1} + \right. \\ & \quad \left. + \left(\int_{-\infty}^{-t/2} + \int_{3t/2}^{\infty} \right) \omega_{\Gamma}(\sigma', |S - 1|) \frac{ds}{\sqrt{s^2 + 1}} \right) \leq \\ & \leq c \|g\|_{C(\overline{\mathbb{R}})} \left(\omega_{\Gamma}(\sigma', 6\varepsilon) \left(\int_{-\infty}^{-t/2} + \int_{3t/2}^{\infty} \right) \frac{t ds}{s^2} + \right. \\ & \quad \left. + \left(\int_{-\infty}^{-t/2} + \int_{3t/2}^{\infty} \right) \frac{\omega_{\Gamma}(\sigma', |S - 1|)}{|S - 1|} \frac{|s| ds}{(s^2 + 1)^{3/2}} \right) \leq \end{aligned}$$

$$\begin{aligned}
&\leq c \|g\|_{C(\mathbb{R})} \left(\omega_{\Gamma}(\sigma', 6\varepsilon) + \int_0^{4\varepsilon} \frac{\omega_{\Gamma}(\sigma', \eta)}{\eta} d\eta \right) \leq c \|g\|_{C(\mathbb{R})} \varepsilon \int_0^2 \frac{\omega_{\Gamma}(\sigma', \eta)}{\eta(\eta + \varepsilon)} d\eta, \\
I_2 &\leq c \|g\|_{C(\mathbb{R})} \int_{t/2}^{3t/2} \left(\frac{\omega_{\Gamma}(\sigma', |S - T|)}{|S - T|} \frac{1 + |s||t|}{\sqrt{s^2 + 1}\sqrt{t^2 + 1}} + \right. \\
&\quad \left. + \omega_{\Gamma}(\sigma', |S - 1|) s \right) \frac{ds}{s^2 + 1} \leq \\
&\leq c \|g\|_{C(\mathbb{R})} \left(\int_0^{2\varepsilon} \frac{\omega_{\Gamma}(\sigma', \eta)}{\eta} d\eta + \omega_{\Gamma}(\sigma', 4\varepsilon) \right) \leq c \|g\|_{C(\mathbb{R})} \varepsilon \int_0^2 \frac{\omega_{\Gamma}(\sigma', \eta)}{\eta(\eta + \varepsilon)} d\eta.
\end{aligned}$$

Here and below in the proof, by c we denote constants whose values are independent of t and ε , but, generally speaking, may be different even within a single chain of inequalities.

For estimating the integral I_3 we use the equality

$$\begin{aligned}
m_1(t, s) - m_1(\infty, s) &= \left(\frac{1}{s - t} - \frac{s}{s^2 + 1} \right) \frac{\sigma'(S) - d(S, T)}{d(S, T)} + \\
&\quad + \frac{s}{s^2 + 1} \frac{\sigma'(S) - d(S, 1)}{d(S, 1)} = \\
&= \frac{s}{s^2 + 1} \sigma'(S) \frac{d(S, T) - d(S, 1)}{d(S, 1) d(S, T)} + \frac{1}{s - t} \frac{\sigma'(S) - d(S, T)}{d(S, T)},
\end{aligned}$$

and the inequalities (6.2), (6.3) and properties of a modulus of continuity (cf., e.g., [26, p. 176]). Thus, we obtain

$$\begin{aligned}
I_3 &\leq c \|g\|_{C(\mathbb{R})} \left(|T - 1| \int_{-t/2}^{t/2} \frac{\omega_{\Gamma}(\sigma', |S - 1|)}{|S - 1|} \frac{|s|}{s^2 + 1} ds + \right. \\
&\quad \left. + \int_{-t/2}^{t/2} \frac{\omega_{\Gamma}(\sigma', |S - T|)}{|S - T|} \frac{ds}{\sqrt{s^2 + 1}\sqrt{t^2 + 1}} \right) \leq
\end{aligned}$$

$$\begin{aligned}
&\leq c \|g\|_{C(\overline{\mathbb{R}})} |T-1| \int_{-t/2}^{t/2} \frac{\omega_{\Gamma}(\sigma', |S-1|)}{|S-1|^2} \frac{|s| ds}{(s^2+1)^{3/2}} \leq \\
&\leq c \|g\|_{C(\overline{\mathbb{R}})} \varepsilon \int_{\varepsilon}^2 \frac{\omega_{\Gamma}(\sigma', \eta)}{\eta^2} d\eta \leq c \|g\|_{C(\overline{\mathbb{R}})} \varepsilon \int_0^2 \frac{\omega_{\Gamma}(\sigma', \eta)}{\eta(\eta+\varepsilon)} d\eta.
\end{aligned}$$

An estimate of the type (6.12) for $I[g, m_1](t)$ follows from the obtained inequalities. An estimate of the type (6.12) for $I[g, m_2](t)$ is similarly established. Thus, the inequality (6.12) is proved for $j = 1$.

To prove the estimate (6.12) for $j = 2$, we use the representation (6.10) of the function $k_2(t, s)$ and obtain estimates of the type (6.12) for each of the integrals $I[g, n_1](t)$, $I[g, n_2](t)$ similarly to the estimation of $I[g, m_1](t)$. The lemma is proved.

The next statement follows obviously from Lemmas 6.4 and 6.11.

Theorem 6.13 *Let the conformal mapping $\sigma(T)$ have the nonvanishing continuous contour derivative $\sigma'(T)$ on the circle Γ , and its modulus of continuity satisfy the condition (6.5). Let the function $k(t, s)$ be defined by the relation (5.7), where the functions $k_1(t, s)$, $k_2(t, s)$ are defined by the equalities (5.8), (5.9), respectively. Then the operator*

$$\mathcal{J}[g] := \int_{-\infty}^{\infty} g(s) k(t, s) ds$$

is compact in the space $C(\overline{\mathbb{R}})$.

7 Equivalence conditions of the (1-3)-problem to a system of Fredholm integral equations

From the above, one can see that finding a solution of the (1-3)-problem in the form (5.1) with functions $\varphi_1 : \partial D_{\zeta} \rightarrow \mathbb{R}$, $\varphi_3 : \partial D_{\zeta} \rightarrow \mathbb{R}$ satisfying conditions of the type (4.3) is reduced to solving the system of integral equations (5.10). Under conditions of Theorem 6.13, integral operators, which are generated by the left parts of system (5.10), are compact in the space $C(\overline{\mathbb{R}})$, i.e. the system (5.10) is a system of Fredholm integral equations.

For any function $g \in C(\overline{\mathbb{R}})$ we use the local centered (with respect to the infinitely remote point) modulus of continuity

$$\omega_{\mathbb{R},\infty}(g, \varepsilon) := \sup_{t \in \mathbb{R} : |t| \geq 1/\varepsilon} |g(t) - g(\infty)|.$$

Let $\mathcal{D}(\overline{\mathbb{R}})$ denote the class of functions $g \in C(\overline{\mathbb{R}})$ whose moduli of continuity satisfy the Dini conditions

$$\int_0^1 \frac{\omega_{\mathbb{R}}(g, \eta)}{\eta} d\eta < \infty, \quad \int_0^1 \frac{\omega_{\mathbb{R},\infty}(g, \eta)}{\eta} d\eta < \infty. \quad (7.1)$$

Since the functions $\varphi_1: \partial D_\zeta \rightarrow \mathbb{R}$ and $\varphi_3: \partial D_\zeta \rightarrow \mathbb{R}$ in the expression (5.1) of a solution of the (1-3)-problem have to satisfy conditions of the type (4.3), it is necessary to require that corresponding functions g_1, g_3 satisfying the system (5.10) should belong to the class $\mathcal{D}(\overline{\mathbb{R}})$. In the next theorem we state a condition on the conformal mapping $\sigma(T)$, under which all solutions of the system (5.10) satisfy the mentioned requirement.

Theorem 7.2 *Let the functions $u_1: \partial D_\zeta \rightarrow \mathbb{R}$, $u_3: \partial D_\zeta \rightarrow \mathbb{R}$ satisfy conditions of the type (4.3). Let the conformal mapping $\sigma(T)$ have the nonvanishing continuous contour derivative $\sigma'(T)$ on the circle Γ , and its modulus of continuity satisfy the condition*

$$\int_0^2 \frac{\omega_\Gamma(\sigma', \eta)}{\eta} \ln \frac{3}{\eta} d\eta < \infty. \quad (7.3)$$

Then all continuous functions g_1, g_3 satisfying the system of Fredholm integral equations (5.10) belong to the class $\mathcal{D}(\overline{\mathbb{R}})$, and the corresponding functions φ_1, φ_3 in (5.1) satisfy conditions of the type (4.3).

Proof. Let us rewrite the system (5.10) in the matrix form:

$$\begin{pmatrix} g_1(t) \\ g_3(t) \end{pmatrix} = \begin{pmatrix} 2\tilde{u}_1(t) \\ 2\tilde{u}_3(t) \end{pmatrix} - \begin{pmatrix} U_1 \left[\frac{1}{\pi i} I[g, k](t) \right] \\ U_3 \left[\frac{1}{\pi i} I[g, k](t) \right] \end{pmatrix} \quad \forall t \in \mathbb{R}, \quad (7.4)$$

where $I[g, k](t) := \int_{-\infty}^{\infty} g(s)k(t, s) ds$ and the function g is defined by the equality (5.4).

Inasmuch as the derivative $\sigma'(T)$ is continuous on Γ and the functions u_1, u_3 satisfy conditions of the type (4.3), the right-hand sides \tilde{u}_1, \tilde{u}_3 of the equations (5.10) belong to the class $\mathcal{D}(\overline{\mathbb{R}})$. With using Lemmas 6.4 and 6.11, it is easy to establish that moduli of continuity of the function $I[g, k](t)$ satisfy conditions of the type (7.1) due to the condition (7.3). Therefore, in view of (7.4) the functions g_1, g_3 belong to the class $\mathcal{D}(\overline{\mathbb{R}})$ also. Finally, by Lemma 3.3 in [27], we conclude that the functions φ_1, φ_3 satisfy conditions of the type (4.3). The theorem is proved.

Let us make some remarks concerning the representation of a solution of the (1-3)-problem by the formula (5.1).

Suppose that a solution Φ of the (1-3)-problem is continuously extended to the boundary ∂D_ζ . By the (2-4)-problem *conjugated* with the (1-3)-problem or, briefly, the *conjugated* (2-4)-problem we shall call a problem on finding a continuous function $\Phi_*: \mu \setminus D_\zeta \longrightarrow \mathbb{B}$ which is monogenic in the domain $\mu \setminus \overline{D}_\zeta$ and vanishes at the infinity with the following boundary conditions:

$$U_2[\Phi_*(\zeta)] = U_2[\Phi(\zeta)], \quad U_4[\Phi_*(\zeta)] = U_4[\Phi(\zeta)] \quad \forall \zeta \in \partial D_\zeta.$$

Note that if a solution of the (1-3)-problem is expressed by the formula (5.1), where the functions φ_1, φ_3 satisfy conditions of the type (4.3), then by Theorem 4.2, the integral (4.1) with $\varphi(\tau) = \varphi_1(\tau)e_1 + \varphi_3(\tau)e_2$, being a solution of the (1-3)-problem, can be continuously extended to the boundary ∂D_ζ from each of the domains $D_\zeta, \mu \setminus \overline{D}_\zeta$ and is also a solution of the conjugated (2-4)-problem due to the formulas (4.4).

Furthermore, by virtue of assumptions that a solution Φ of the (1-3)-problem is continuously extended to the boundary ∂D_ζ and the conjugated (2-4)-problem has a solution Φ_* , it follows that the function Φ is represented in the form (5.1). Indeed, by the Cauchy integral formula and the Cauchy theorem for monogenic functions in the biharmonic plane (cf., e.g., Theorem 3.2 in [28]), the following equalities are true:

$$\Phi(\zeta) = \frac{1}{2\pi i} \int_{\partial D_\zeta} \Phi(\tau)(\tau - \zeta)^{-1} d\tau, \quad 0 = \frac{1}{2\pi i} \int_{\partial D_\zeta} \Phi_*(\tau)(\tau - \zeta)^{-1} d\tau \quad \forall \zeta \in D_\zeta,$$

that implies the equality (5.1) with

$$\varphi_j(\tau) = U_j[\Phi(\tau)] - U_j[\Phi_*(\tau)] = u_j(\tau) - U_j[\Phi_*(\tau)], \quad j \in \{1, 3\}.$$

The made remarks can be amplified by the following theorem.

Theorem 7.5 *Let the functions $u_1 : \partial D_\zeta \rightarrow \mathbb{R}$, $u_3 : \partial D_\zeta \rightarrow \mathbb{R}$ satisfy conditions of the type (4.3). Let the conformal mapping $\sigma(T)$ have the non-vanishing continuous contour derivative $\sigma'(T)$ on the circle Γ , and its modulus of continuity satisfy the condition (7.3). Then the following assertions are equivalent:*

- (I) *the system of Fredholm integral equations (5.10) is solvable in the space $C(\overline{\mathbb{R}})$;*
- (II) *there exists a solution of the (1-3)-problem of the form (5.1), where the functions φ_1 , φ_3 satisfy conditions of the type (4.3);*
- (III) *a solution Φ of the (1-3)-problem is continuously extended to the boundary ∂D_ζ . For this function Φ , the conjugated (2-4)-problem is solvable and moduli of continuity of components $U_1[\Phi_*]$, $U_3[\Phi_*]$ of its solution Φ_* satisfy conditions of the type (4.3).*

Proof. Continuing the reasonings adduced before Theorem 7.5, we conclude that in the case where the functions u_1 , u_3 satisfy conditions of the type (4.3), the functions φ_1 , φ_3 satisfy the same conditions if and only if moduli of continuity of components $U_1[\Phi_*]$, $U_3[\Phi_*]$ of a solution Φ_* of the conjugated (2-4)-problem satisfy conditions of the type (4.3). Thus, the assertions (II) and (III) are equivalent. The equivalence of assertions (I) and (II) is a consequence of Theorem 7.2. The theorem is proved.

Rewrite the integral equations of the system (5.10) in expanded form:

$$\begin{aligned}
& \frac{1}{2} g_1(t) + \frac{1}{2\pi} \operatorname{Im} \int_{-\infty}^{\infty} g_1(s) \frac{\tau'(s)}{\tau(s) - \tau(t)} ds + \frac{1}{2\pi} \operatorname{Re} \int_{-\infty}^{\infty} g_1(s) \frac{\tau'_1(s)(\tau_2(s) - \tau_2(t))}{(\tau(s) - \tau(t))^2} ds - \\
& - \frac{1}{2\pi} \operatorname{Re} \int_{-\infty}^{\infty} g_1(s) \frac{\tau'_2(s)(\tau_1(s) - \tau_1(t))}{(\tau(s) - \tau(t))^2} ds - \\
& - \frac{1}{2\pi} \operatorname{Im} \int_{-\infty}^{\infty} g_3(s) \frac{\tau'_1(s)(\tau_2(s) - \tau_2(t))}{(\tau(s) - \tau(t))^2} ds + \\
& + \frac{1}{2\pi} \operatorname{Im} \int_{-\infty}^{\infty} g_3(s) \frac{\tau'_2(s)(\tau_1(s) - \tau_1(t))}{(\tau(s) - \tau(t))^2} ds = \tilde{u}_1(t),
\end{aligned}$$

$$\begin{aligned}
& \frac{1}{2} g_3(t) + \frac{1}{2\pi} \operatorname{Im} \int_{-\infty}^{\infty} g_3(s) \frac{\tau'(s)}{\tau(s) - \tau(t)} ds - \frac{1}{2\pi} \operatorname{Re} \int_{-\infty}^{\infty} g_3(s) \frac{\tau'_1(s)(\tau_2(s) - \tau_2(t))}{(\tau(s) - \tau(t))^2} ds + \\
& + \frac{1}{2\pi} \operatorname{Re} \int_{-\infty}^{\infty} g_3(s) \frac{\tau'_2(s)(\tau_1(s) - \tau_1(t))}{(\tau(s) - \tau(t))^2} ds - \\
& - \frac{1}{2\pi} \operatorname{Im} \int_{-\infty}^{\infty} g_1(s) \frac{\tau'_1(s)(\tau_2(s) - \tau_2(t))}{(\tau(s) - \tau(t))^2} ds + \\
& + \frac{1}{2\pi} \operatorname{Im} \int_{-\infty}^{\infty} g_1(s) \frac{\tau'_2(s)(\tau_1(s) - \tau_1(t))}{(\tau(s) - \tau(t))^2} ds = \tilde{u}_3(t) \quad \forall t \in \mathbb{R}.
\end{aligned}$$

Then the homogeneous system transposed with the system (5.10) is of the form:

$$\begin{aligned}
h_1(t) - \frac{1}{\pi} \operatorname{Im} \left(\tau'(t) \int_{-\infty}^{\infty} \frac{h_1(s) ds}{\tau(s) - \tau(t)} \right) - \frac{\tau'_1(t)}{\pi} \operatorname{Re} \int_{-\infty}^{\infty} h_1(s) \frac{(\tau_2(s) - \tau_2(t))}{(\tau(s) - \tau(t))^2} ds + \\
+ \frac{\tau'_2(t)}{\pi} \operatorname{Re} \int_{-\infty}^{\infty} h_1(s) \frac{(\tau_1(s) - \tau_1(t))}{(\tau(s) - \tau(t))^2} ds + \frac{\tau'_1(t)}{\pi} \operatorname{Im} \int_{-\infty}^{\infty} h_3(s) \frac{(\tau_2(s) - \tau_2(t))}{(\tau(s) - \tau(t))^2} ds - \\
- \frac{\tau'_2(t)}{\pi} \operatorname{Im} \int_{-\infty}^{\infty} h_3(s) \frac{(\tau_1(s) - \tau_1(t))}{(\tau(s) - \tau(t))^2} ds = 0, \tag{7.6}
\end{aligned}$$

$$\begin{aligned}
h_3(t) - \frac{1}{\pi} \operatorname{Im} \left(\tau'(t) \int_{-\infty}^{\infty} \frac{h_3(s) ds}{\tau(s) - \tau(t)} \right) + \frac{\tau'_1(t)}{\pi} \operatorname{Re} \int_{-\infty}^{\infty} h_3(s) \frac{(\tau_2(s) - \tau_2(t))}{(\tau(s) - \tau(t))^2} ds - \\
- \frac{\tau'_2(t)}{\pi} \operatorname{Re} \int_{-\infty}^{\infty} h_3(s) \frac{(\tau_1(s) - \tau_1(t))}{(\tau(s) - \tau(t))^2} ds + \frac{\tau'_1(t)}{\pi} \operatorname{Im} \int_{-\infty}^{\infty} h_1(s) \frac{(\tau_2(s) - \tau_2(t))}{(\tau(s) - \tau(t))^2} ds - \\
- \frac{\tau'_2(t)}{\pi} \operatorname{Im} \int_{-\infty}^{\infty} h_1(s) \frac{(\tau_1(s) - \tau_1(t))}{(\tau(s) - \tau(t))^2} ds = 0 \quad \forall t \in \mathbb{R}. \tag{7.7}
\end{aligned}$$

Lemma 7.8 *The pair of functions $(h_1, h_2) := (\tau'_1, \tau'_2)$ satisfies the system of integral equations (7.6), (7.7).*

Proof. Substituting $h_1 = \tau'_1$, $h_2 = \tau'_2$ into the equation (7.6) and applying the conformal mapping $z = \tau(t)$ of the upper half-plane $\{t \in \mathbb{C} : \text{Im } t > 0\}$ onto the domain D_z of complex plane, we pass in (7.6) to integrating along the boundary ∂D_z of domain D_z . In such a way, using the denotations $v_1 := \tau_1(s)$, $v_2 := \tau_2(s)$, $v := \tau(s) \equiv v_1 + iv_2$, $x := \tau_1(t)$, $y := \tau_2(t)$, $z \equiv x + iy$, we obtain

$$\begin{aligned} & \tau'_1(t) - \frac{1}{\pi} \text{Im} \left(\tau'(t) \int_{\partial D_z} \frac{dv_1}{v - z} \right) - \\ & - \frac{\tau'_1(t)}{\pi} \text{Re} \int_{\partial D_z} \frac{v_2 - y}{(v - z)^2} dv_1 + \frac{\tau'_2(t)}{\pi} \text{Re} \int_{\partial D_z} \frac{v_1 - x}{(v - z)^2} dv_1 + \\ & + \frac{\tau'_1(t)}{\pi} \text{Im} \int_{\partial D_z} \frac{v_2 - y}{(v - z)^2} dv_2 - \frac{\tau'_2(t)}{\pi} \text{Im} \int_{\partial D_z} \frac{v_1 - x}{(v - z)^2} dv_2 = 0. \end{aligned} \quad (7.9)$$

Using polar coordinates (r, θ) with the relation $v - z =: r \exp\{i\theta\}$, it is easy to show that the equality (7.9) becomes identical.

In a similar way, we conclude that the functions $h_1 = \tau'_1$, $h_2 = \tau'_2$ satisfy the equation (7.7). The lemma is proved.

It follows from Lemma 7.8 that the condition

$$\int_{-\infty}^{\infty} \left(\tilde{u}_1(s) \tau'_1(s) + \tilde{u}_3(s) \tau'_2(s) \right) ds = 0$$

is necessary for the solvability of the system of integral equations (5.10). Passing in this equality to the integration along the boundary ∂D_ζ , we get the equivalent condition

$$\int_{\partial D_\zeta} u_1(xe_1 + ye_2) dx + u_3(xe_1 + ye_2) dy = 0 \quad (7.10)$$

for given functions u_1, u_3 .

It is proved in [14] that the condition (7.10) is also sufficient for the solvability of the (1-3)-problem for a disk. The next theorem contains assumptions, under which the condition (7.10) is necessary and sufficient for the solvability of the system of integral equations (5.10) and, therefore, for the existence of a solution of the (1-3)-problem in the form (5.1).

Theorem 7.11 *Assume that the conformal mapping $\sigma(T)$ has the nonvanishing continuous contour derivative $\sigma'(T)$ on the circle Γ , and its modulus of continuity satisfies the condition (7.3). Also, assume that all solutions g_1, g_3 of the homogeneous system of equations (5.10) (with $\tilde{u}_j \equiv 0$ for $j \in \{1, 3\}$) are differentiable on \mathbb{R} , and the integral*

$$\frac{1}{2\pi i} \int_{\partial D_\zeta} \varphi'(\tau)(\tau - \zeta)^{-1} d\tau$$

is bounded in both domains D_ζ and $\mu \setminus \overline{D_\zeta}$; here φ' is the contour derivative of the function $\varphi(\tau) := g_1(s)e_1 + g_3(s)e_2$, where $\tau = \tilde{\tau}(s)$ for all $s \in \mathbb{R}$. Then the following assertions are true:

- 1) *the number of linearly independent solutions of the homogeneous system of equations (5.10) is equal to 1;*
- 2) *the non-homogeneous system of equations (5.10) is solvable if and only if the condition (7.10) is satisfied.*

Proof. It follows from Lemma 7.8 that the transposed system of equations (7.6), (7.7) has at least one nontrivial solution. Therefore, by the Fredholm theory, the system of homogeneous equations (5.10) has at least one nontrivial solution $g_1 = g_1^0, g_3 = g_3^0$. Consider the function

$$\varphi_0(\tau) := g_1^0(s)e_1 + g_3^0(s)e_2, \quad \tau = \tilde{\tau}(s), \quad \forall s \in \mathbb{R} \quad (7.12)$$

corresponding to this solution.

Then the function, which is defined for all $\zeta \in D_\zeta$ by the formula

$$\Phi_0(\zeta) := \frac{1}{2\pi i} \int_{\partial D_\zeta} \varphi_0(\tau)(\tau - \zeta)^{-1} d\tau, \quad (7.13)$$

is a solution of the homogeneous (1-3)-problem (with $u_1 = u_3 \equiv 0$). By virtue of differentiability of the functions g_1^0, g_3^0 on \mathbb{R} , the function φ_0 has the contour derivative $\varphi_0'(\tau)$ for all $\tau \in \partial D_\zeta$ and the following equality holds:

$$\Phi_0'(\zeta) := \frac{1}{2\pi i} \int_{\partial D_\zeta} \varphi_0'(\tau)(\tau - \zeta)^{-1} d\tau \quad \forall \zeta \in D_\zeta. \quad (7.14)$$

Let Φ_1 be a monogenic in D_ζ function such that $\Phi_1'(\zeta) = \Phi_0'(\zeta)$ for all $\zeta \in D_\zeta$. Then the function $V(x, y) := U_1[\Phi_1(\zeta)]$ is a solution of the homogeneous biharmonic problem (1.2) (with $\omega_3 = \omega_4 \equiv 0$).

Taking into account the equalities

$$\begin{aligned} \Delta V(x, y) &:= \frac{\partial^2 V(x, y)}{\partial x^2} + \frac{\partial^2 V(x, y)}{\partial y^2} = \\ &= U_1 [\Phi_1''(\zeta)(e_1^2 + e_2^2)] = U_1 [\Phi_0'(\zeta)(e_1^2 + e_2^2)] \end{aligned}$$

and boundedness of the function (7.14) in the domain D_ζ , we conclude that the Laplacian $\Delta V(x, y)$ is bounded in the domain D .

By the uniqueness theorem (see § 4 in [29]), solutions of the homogeneous biharmonic problem (1.2) in the class of functions V , for which $\Delta V(x, y)$ is bounded in D , are only constants: $V = \text{const}$. Therefore, taking into account the fact that the functions (7.13) and (3.1) are equal, we obtain the equalities $U_1[\Phi_0(\zeta)] = U_3[\Phi_0(\zeta)] \equiv 0$ in the domain D_ζ .

All monogenic functions Φ_0 satisfying the condition $U_1[\Phi_0(\zeta)] \equiv 0$ are described in [13, Lemma 3]. Taking into account the identity $U_3[\Phi_0(\zeta)] \equiv 0$ also, we obtain the equality

$$\Phi_0(\zeta) = ik\zeta + i(n_1 e_1 + n_2 e_2) \quad \forall \zeta \in \overline{D_\zeta}, \quad (7.15)$$

where k, n_1 and n_2 are real numbers.

Let us show that the constant k does not equal to zero in the equality (7.15). Assume the contrary: $k = 0$. Then it follows from the Sokhotski-Plemelj formula (4.4) for the function (7.13) and the identity $\Phi_0^+(\tau) \equiv i(n_1 e_1 + n_2 e_2)$ that the following equality holds:

$$\Phi_0^-(\tau) = i(n_1 e_1 + n_2 e_2) - \varphi_0(\tau) \quad \forall \tau \in \partial D_\zeta. \quad (7.16)$$

Using the integral Cauchy formula for monogenic functions in the biharmonic plane (cf., e.g., Theorem 3.2 in [28]) and taking into account the

equality (7.16), we obtain the equalities

$$\begin{aligned}
\Phi_0(\zeta) &= -\frac{1}{2\pi i} \int_{\partial D_\zeta} \Phi_0^-(\tau)(\tau - \zeta)^{-1} d\tau = \\
&= \frac{1}{2\pi i} \int_{\partial D_\zeta} \varphi_0(\tau)(\tau - \zeta)^{-1} d\tau - \frac{i(n_1 e_1 + n_2 e_2)}{2\pi i} \int_{\partial D_\zeta} (\tau - \zeta)^{-1} d\tau = \\
&= \Phi_0(\zeta) - i(n_1 e_1 + n_2 e_2) \quad \forall \zeta \in \mu \setminus \overline{D_\zeta},
\end{aligned}$$

a corollary of which is the equality $n_1 e_1 + n_2 e_2 = 0$. In virtue of the uniqueness of decomposition of any element in the algebra \mathbb{B} with respect to the basis $\{e_1, e_2\}$, we get $n_1 = n_2 = 0$.

Thus, the equality $\Phi_0(\zeta) = 0$ for all $\zeta \in \overline{D_\zeta}$ is a corollary of the assumption $k = 0$. Furthermore, the function (7.13), which is considered for all $\zeta \in \mu \setminus \overline{D_\zeta}$, is a solution of the conjugated (2-4)-problem with boundary data:

$$U_2[\Phi_0^-(\zeta)] = U_2[\Phi_0^+(\zeta)] \equiv 0, \quad U_4[\Phi_0^-(\zeta)] = U_4[\Phi_0^+(\zeta)] \equiv 0 \quad \forall \zeta \in \partial D_\zeta.$$

Using arguments analogous to those used in the proof of the equality (7.15), we obtain the equality

$$\Phi_0(\zeta) = k_1 \zeta + m_1 e_1 + m_2 e_2 \quad \forall \zeta \in \mu \setminus \overline{D_\zeta},$$

where k_1 , m_1 and m_2 are real numbers. Therefore, as a corollary of (7.16), we get the equality

$$\varphi_0(\tau) = -\Phi_0^-(\tau) = -k_1 \tau - m_1 e_1 - m_2 e_2 \quad \forall \tau \in \partial D_\zeta. \quad (7.17)$$

Substituting the expression (7.17) of the function φ_0 in (7.13) and taking into account the equality $\Phi_0(\zeta) = 0$ for all $\zeta \in D_\zeta$, we obtain the equality

$$0 = -k_1 \zeta - m_1 e_1 - m_2 e_2 \quad \forall \zeta \in D_\zeta,$$

whence we get $k_1 = m_1 = m_2 = 0$. As a result, the equality (7.17) is reduced to the identity $\varphi_0(\tau) \equiv 0$ which contradicts to the non-triviality of at least one of the functions g_1^0, g_3^0 in the definition (7.12) of the function φ_0 .

Hence, the assumption $k = 0$ is not true. Therefore, the equality (7.15) holds with $k \neq 0$.

Let $g_1 = g_1^1$, $g_3 = g_3^1$ be a nontrivial solution of the homogeneous system (5.10), and g_1^1, g_3^1 be different from g_1^0, g_3^0 . Then for the function $\varphi^1(\tau) := g_1^1(s)e_1 + g_3^1(s)e_2$, where $\tau = \tilde{\tau}(s)$ for all $s \in \mathbb{R}$, the following equality of type (7.15) holds:

$$\frac{1}{2\pi i} \int_{\partial D_\zeta} \varphi^1(\tau)(\tau - \zeta)^{-1} d\tau = i\tilde{k}\zeta + i(\tilde{n}_1 e_1 + \tilde{n}_2 e_2) \quad \forall \zeta \in \overline{D_\zeta}, \quad (7.18)$$

where \tilde{k} , \tilde{n}_1 and \tilde{n}_2 are real numbers.

Define a function $\tilde{\varphi}$ by the equality

$$\tilde{\varphi}(\tau) = \varphi^1(\tau) - \frac{\tilde{k}}{k} \varphi_0(\tau) \quad \forall \tau \in \partial D_\zeta.$$

Taking into account the equalities (7.15) and (7.18), we obtain

$$\frac{1}{2\pi i} \int_{\partial D_\zeta} \tilde{\varphi}(\tau)(\tau - \zeta)^{-1} d\tau = ie_1 \left(\tilde{n}_1 - n_1 \frac{\tilde{k}}{k} \right) + ie_2 \left(\tilde{n}_2 - n_2 \frac{\tilde{k}}{k} \right) \quad \forall \zeta \in \overline{D_\zeta}.$$

Further, in such a way as for the assumption $k = 0$ in the equality (7.15), we obtain the identity $\tilde{\varphi}(\tau) \equiv 0$ which implies the equalities $g_m^1(\tau) = \frac{\tilde{k}}{k} g_m^0(\tau)$ for all $\tau \in \partial D_\zeta$ and $m \in \{1, 3\}$. The assertion 1) is proved.

Due to the assertion 1), by the Fredholm theory, the number of linearly independent solutions of the transposed system of equations (7.6), (7.7) is also equal to 1. Such a nontrivial solution is described in Lemma 7.8. Therefore, the condition (7.10) is not only necessary but also sufficient for the solvability of the system of integral equations (5.10). The theorem is proved.

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